ON NEW KINDS OF TEICHMÜLLER SPACES†

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ABSTRACT

We introduce a Teichmüller space for a Riemann surface with n distinguished points. If $n = 0$ this is the ordinary Teichmüller space. For $n = 1$, in special cases, it represents the Teichmüller curve and the universal covering space of the Teichmüller curve. The corresponding modular groups and Riemann spaces are investigated. Some purely topological applications on homotopy of self-maps of surfaces are obtained.

In this paper we outline an extension of the theory of Teichmüller spaces as developed by Ahlfors [2] and Bers [8]. The extension of the theory allows us to generalize as well as simplify the proofs of the results of Bers [9] on the space of moduli of a Riemann surface. We obtain descriptions of the mapping class groups of a Riemann surface similar to those obtained by Birman [11], [12]. We also obtain some purely topological applications concerning homotopy classes of topological self-mappings of Riemann surfaces.

We shall describe our results for Riemann surfaces of finite type. The extension to the more general situation as well as complete proofs will appear elsewhere.

This paper is, in a sense, foundational. It unifies various concepts that have often appeared in the literature. For example, the Teichmüller space, the universal curve over the Teichmüller space, as well as the covering space of the universal curve, are all special cases of the same generalized Teichmüller space that we introduce. Many problems are suggested by this generalization. We will list some of these, indicate their current state, and continue to investigate them in subsequent papers.

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In addition to the papers of Bers [9] and Birman [11], [12], this work touches (tangentially at times) investigations of Engber [17], Epstein [18], Earle [14] Marden [20], Bers-Greenberg [10], among others.

1. Teichmüller spaces of n-pointed Fuchsian groups

Let U be the upper half plane, and Aut U the group of conformal self-mappings of U:

$$\mathrm{Aut}\, U \cong SL(2,\mathbb{R}) / \pm I$$

or

$$g \in \mathrm{Aut}\, U \Leftrightarrow g(z) = \frac{az+b}{cz+d}, \quad \text{for } z \in U;\ a,b,c,d \in \mathbb{R};\ ad-bc = 1.$$

(Here $\mathbb{R}$ is the real line, $SL(2,\mathbb{R})$ the group of 2×2 real matrices with determinant 1, and I the 2×2 identity matrix.)

Let $G \subset \mathrm{Aut}\, U$ be a finitely generated Fuchsian group of the first kind. (Hereafter, unless otherwise indicated, *all Fuchsian groups are assumed to be finitely generated of the first kind and operating on U.*)

Let $M(G)$ denote the set of Beltrami coefficients for G (open subset in a Banach space); that is, $\mu \in M(G)$ is an equivalence class of measurable functions (modulo functions vanishing a.e.) such that

(i) $\mu(gz)\overline{g'(z)}/g'(z) = \mu(z)$, all $g \in G$, a.e. $z \in U$, and

(ii) $\|\mu\| = \sup\{|\mu(z)|;\ z \in U\} < 1$.

For $\mu \in M(G)$, let w_μ be the unique normalized μ-conformal self-mapping of U which fixes $0, 1, \infty$ (see Ahlfors-Bers [3]), and χ_μ the isomorphism of G onto the Fuchsian group $G_\mu = w_\mu G w_\mu^{-1}$ defined by

$$\chi_\mu(g) = w_\mu \circ g \circ w_\mu^{-1}, \text{ for } g \in G.$$

We shall say that $\mathcal{G} = \{G; z_1, \cdots, z_n\}$ is an *n-pointed Fuchsian group*, provided G is a Fuchsian group and $z_1, \cdots, z_n$ are n-inequivalent points of U which are not fixed points of any elliptic element of G. Of course, the 0-pointed group $\mathcal{G} = \{G; -\}$ will be identified with G. Let

$$U_{\mathcal{G}} = \{z \in U;\ z \text{ is not an elliptic fixed point of } G$$
$$\text{and } z \neq g(z_j) \text{ for all } g \in G,\ j = 1, \cdots, n\}.$$

The Riemann surface $U_{\mathscr{G}}/G$ is a compact surface of genus g from which m points have been removed with $m \geqq n$.

Let

$$\pi: U \to U/G$$

be the natural projection, and note that π is unramified over U_G/G. Let $p_1, \cdots, p_m$ be the punctures on $U_{\mathscr{G}}/G$, and set v_i to be the order of the stability subgroup of $\pi^{-1}(p_i)$. ($v_i = \infty$ if $p_i \notin U/G$.) The $(m+1)$-tuple

$$(g; v_1, \cdots, v_m)$$

is called the *signature* of $\mathscr{G}$. Note that we may assume, and always do, that

$$1 \leqq v_1 \leqq v_2 \leqq \cdots \leqq v_m \leqq \infty.$$

The triple $(g, n, m-n)$ will be called the *type* of $\mathscr{G}$.

Let us call two n-pointed Fuchsian groups $\mathscr{G} = \{G; z_1, \cdots, z_n\}$ and $\mathscr{G}' = \{G'; z'_1, \cdots, z'_n\}$ *conjugate* if there exists an $\alpha \in \operatorname{Aut} U$ such that

$$G' = \alpha G \alpha^{-1},$$

and for $j = 1, \cdots, n$

$$z'_j = \alpha(g_j z_{\sigma(j)}), \text{ with } g_j \in G,$$

where σ is a permutation of $\{1, \cdots, n\}$.

Let K be a subgroup of G. For $\mu, v \in M(G)$, we shall say that μ is $(\mathscr{G}, K)$-*equivalent* to μ (writing either $\mu \sim v(\mathscr{G}, K)$ or $\mu \sim v$) provided

(i) $\chi_\mu = \chi_v$, and
(ii) $w_\mu(z_j)$ is K_μ $(= K_v)$-equivalent to $w_v(z_j)$ $j = 1, \cdots, n$.

Note that, in the future, the pair $(\mathscr{G}, \{1\})$ will be abbreviated as $\mathscr{G}$.

REMARK. Observe that

(1) $$\mu \sim v(\mathscr{G}) \Leftrightarrow w_\mu = w_v \text{ on } \mathbb{R} \cup \{g(z_j); g \in G, \ j = 1, \cdots, n\}.$$

DEFINITION 1.1. The set of equivalence classes of Beltrami coefficients under the equivalence relation above is the *Teichmüller space of $\mathscr{G}$ modulo K*, $T(\mathscr{G}, K)$. It is topologized and given in analytic structure by requiring the projection

$$\Phi_{\mathscr{G},K} = \Phi: M(G) \to T(\mathscr{G}, K)$$

to be continuous and holomorphic.

REMARK. We have the obvious projections which are complex analytic:

$$T(\mathscr{G}) \to T(\mathscr{G}, K) \to T(\mathscr{G}, G) \to T(G).$$

CONVENTION. We shall restrict our attention to the special case where K is a normal subgroup of G. For, in this case, the pairs

$$(\mathscr{G}, K) \text{ and } (\mathscr{G}', K)$$

(for $\mathscr{G} = \{G; z_1, \cdots, z_n\}$ and $\mathscr{G}' = \{G; g_1(z_{\sigma(1)}), \cdots, g_n(z_{\sigma(n)})\}$, where σ is a permutation of $\{1, \cdots, n\}$ and $g_j \in G$ for $j = 1, \cdots, n$) give the same Teichmüller spaces, and we will hence identify such pairs.

The work of Teichmüller [24], [25], Ahlfors [2], and Bers [8] shows that $T(G)$ is a finite dimensional, complex analytic manifold which is homeomorphic to a Euclidean cell.

For fixed $z \in U$

$$M(G) \ni \mu \mapsto w_\mu(z) \in U$$

is a real analytic map but it is not complex analytic (Ahlfors-Bers [3]). To obtain a corresponding complex analytic map, let w^μ (for $\mu \in M(G)$) be the unique normalized automorphism of the complex sphere that is μ-conformal in U and conformal in the lower half plane, L. Bers [7] shows that

$$\mu \sim \nu(G) \Leftrightarrow w^\mu \big| L = w^\nu \big| L,$$

and Ahlfors-Bers [3] show that

$$M(G) \ni \mu \mapsto w^\mu(z) \in \mathbb{C} \text{ for fixed } z \in \mathbb{C}$$

is a holomorphic function.

Furthermore, the group $G^\mu = (w^\mu)G(w^\mu)^{-1}$ is a quasi-Fuchsian group (a Kleinian group of Möbius transformations) that fixes the interior of the quasi-circle $w^\mu(\mathbb{R})$; the domains $w^\mu(U)$ and $w^\mu(L)$ depend only on $\Phi_G(\mu)$. Also for arbitrary $(\mathscr{G}, K)$, we have that

$$\mu \sim \nu(\mathscr{G}, K) \Leftrightarrow w^\mu \big| L = w^\nu \big| L \text{ for } \mu, \nu \in M(G)$$

and that $w^\mu(z_j)$ is K^μ-equivalent to $w^\nu(z_j)$ for $j = 1, \cdots, n$.

2. Teichmüller spaces of n-pointed Riemann surfaces

By an *n-pointed Riemann surface* $\mathscr{S} = \{S; x_1, \cdots, x_n\}$ we mean a Riemann surface S of finite type (that is, a compact surface with a finite number of punctures), on which we distinguish n distinct points $\{x_1, \cdots, x_n\}$, $n \geqq 0$. (If $n = 0$, we identify $\{S; -\}$ with S.) By a homeomorphism W between two n-pointed surfaces

$\mathscr{S} = \{S; x_1, \cdots, x_n\}$ and $\mathscr{S}' = \{S'; x_1', \cdots, x_n'\}$, we mean a homeomorphism W of S onto S' such that for some permutation σ of $\{1, \cdots, n\}$ we have

$$W(x_j) = x'_{\sigma(j)}, \ j = 1, \cdots, n.$$

Fix an n-pointed Riemann surface, $\mathscr{S} = \{S; x_1, \cdots, x_n\}$. Let w_1 and w_2 be two quasiconformal homeomorphisms

$$w_j: \mathscr{S} \to \mathscr{S}_j \ (j = 1, 2). \tag{2}$$

We shall say that w_1 and w_2 are, respectively, *Teichmüller equivalent* and *weakly equivalent* if there exists a conformal map

$$f: \mathscr{S}_1 \to \mathscr{S}_2$$

such that

$$W = w_2^{-1} \circ f \circ w_1: \mathscr{S} \to \mathscr{S}$$

is homotopic to the identity respectively by a homotopy which keeps $x_1, \cdots, x_n$ fixed or by a homotopy whose ends keep $x_1, \cdots, x_n$ fixed. Note that, for equivalence, the homotopy of W to the identity must keep the distinguished points fixed while, for weak equivalence, W only need keep the distinguished points fixed.

DEFINITION 2.1. The set of Teichmüller (or weak) equivalence classes of quasiconformal homeomorphisms of $\mathscr{S}$ forms the *Teichmüller space* (or *weak Teichmüller space*) of $\mathscr{S}$, in symbols $T(\mathscr{S})$ (or $T_0(\mathscr{S})$).

PROPOSITION 2.2. *Let* $\mathscr{G} = \{G; z_1, \cdots, z_n\}$ *be an n-pointed Fuchsian group with G fixed point free.*

Let

$$\pi: U \to U/G$$

be the natural holomorphic projection, and set

$$\mathscr{S} = \{U/G; \pi(z_1), \cdots, \pi(z_n)\}.$$

We have

$$T(\mathscr{G}) \cong T(\mathscr{S}),$$

and

$$T(\mathscr{G}, G) \cong T_0(\mathscr{S}).$$

3. Modular groups

Fix an n-pointed Fuchsian group $\mathscr{G}$ (where $\mathscr{G} = \{G; z_1, \cdots, z_n\}$), and a normal subgroup K of G. Let w be a quasiconformal self-mapping of U (which is not necessarily normalized) that is compatible with G (that is, wGw^{-1} is once again a Fuchsian group). The mapping w induces an *allowable* mapping

$$w^*: T(\mathscr{G}, K) \to T(w\mathscr{G}w^{-1}, wKw^{-1}),$$

where

$$w\mathscr{G}w^{-1} = \{wGw^{-1}; w(z_1), \cdots, w(z_n)\},$$

and

$$w^*[w_\mu]_{(\mathscr{G},K)} = [\alpha[\mu] \circ w_\mu \circ w^{-1}]_{(w\mathscr{G}w^{-1}, wKw^{-1})}.$$

Here $[w_\mu]_{(\mathscr{G},K)}$ is the $(\mathscr{G}, K)$-equivalence class of $\mu \in M(G)$ and $\alpha[\mu]$ is the uniquely determined Möbius transformation so that $\alpha[\mu] \circ w_\mu \circ w^{-1}$ is normalized (hence a w_ν for some $\nu \in M(wGw^{-1})$).

Definition 3.1. The *modular group* of $(\mathscr{G}, K)$, $\mathrm{Mod}(\mathscr{G}, K)$, is the group of allowable self-mappings of $T(\mathscr{G}, K)$.

Thus, elements of $\mathrm{Mod}(\mathscr{G}, K)$ are induced by quasiconformal automorphisms of U that conjugate $(\mathscr{G}, K)$ into itself. (Recall the identification of such pairs introduced in Section 1.)

Similarly, any quasiconformal homeomorphism

$$W: \mathscr{S} \to \mathscr{S}'$$

between n-pointed Riemann surfaces induces *allowable* mappings

$$W^*: T(\mathscr{S}) \to T(\mathscr{S}')$$

and

$$W^*: T_0(\mathscr{S}) \to T_0(\mathscr{S}')$$

by sending an equivalence class of w_1, as in (2), into the equivalence class of $w_1 \circ W^{-1}$. As above, we obtain groups $\mathrm{Mod}\,\mathscr{S}$ and $\mathrm{Mod}_0\,\mathscr{S}$, respectively, the *Teichmüller modular group of* $\mathscr{S}$ (allowable self-mappings of $T(\mathscr{S})$) and the *weak modular group of* $\mathscr{S}$ (allowable self-mappings of $T_0(\mathscr{S})$). From the identification of Teichmüller spaces given by Proposition 2.2, we have

$$\mathrm{Mod}\,\mathscr{G} \cong \mathrm{Mod}\,\mathscr{S},$$

and

$$\mathrm{Mod}(\mathscr{G}, G) \cong \mathrm{Mod}_0\mathscr{S},$$

whenever G acts freely.

Note that every allowable mapping between Teichmüller spaces is a biholomorphic homeomorphism. Furthermore, if $\mathscr{G}$ and $\mathscr{G}'$ are two pointed groups with the same signature, then there exists a quasiconformal automorphism w of U such that

$$\mathscr{G}' = w\mathscr{G}w^{-1}.$$

DEFINITION 3.2. The Teichmüller spaces and modular groups for signature $(g; \nu_1, \cdots, \nu_m)$ are defined by

$$T(g; \nu_1, \cdots, \nu_m) = T(\mathscr{G}),$$

$$\operatorname{Mod}(g; \nu_1, \cdots, \nu_m) = \operatorname{Mod}\mathscr{G},$$

where $\mathscr{G}$ is a pointed group whose signature is $(g; \nu_1, \cdots, \nu_m)$.

4. Isomorphism theorems

Let $\mathscr{G} = \{G; z_1, \cdots, z_n\}$ be an n-pointed Fuchsian group. Choose a holomorphic universal covering map

$$h: U \to U_{\mathscr{G}}$$

with covering group H. Let Γ be the Fuchsian model of $(G, U_{\mathscr{G}})$ via h; that is,

$$\Gamma = \{\gamma \in \operatorname{Aut} U;\ h \circ \gamma = g \circ h \text{ for some } g \in G\}.$$

(H is, of course, the subgroup of Γ consisting of those γ with $g = 1$.) Hence we have an exact sequence of groups and group homomorphisms

$$\{1\} \to H \to \Gamma \to G \to \{1\}.$$

THEOREM 4.1. *There exists a canonical, complex analytic, surjective mapping*

$$h^*: T(\Gamma) \to T(\mathscr{G}). \tag{3}$$

Furthermore, h^* is injective (thus an isomorphism) provided either that

(i) $\mathscr{G} = G$, *or*

(ii) *G does not contain elliptic elements and $n = 1$.*

In all other cases h^* is a non-trivial (that is, non-injective) universal covering map whose covering group is a fixed point free subgroup of Mod Γ.

We outline the proof of the theorem, since it gives the right setting for a discussion of various other questions. Define a complex linear isomorphism h which is isometric:

$$h\colon M(\Gamma) \to M(G)$$

by setting

$$(4) \qquad (h\mu)\,(hz)\frac{\overline{h'(z)}}{h'(z)} = \mu(z), \text{ for } \mu \in M(\Gamma),\ z \in U.$$

It is quite easy to check that the above definition leads to a commutative diagram as shown in Fig. 1, where h_μ is a (uniquely defined) holomorphic universal covering map of $U_{\mathscr{G}_{h'}} = w_{h\mu}(U_{\mathscr{G}})$, and $\mathscr{G}_{h\mu} = \{G_{h\mu};\, w_{h\mu}(z_1), \cdots, w_{h\mu}(z_n)\}$. One checks that $\mu \sim \nu(\Gamma)$ (for $\mu, \nu \in M(\Gamma)$) implies that $h\mu \sim h\nu(\mathscr{G})$. Thus the map h of (4) projects to a mapping h^* between Teichmüller spaces as required in (3). The mapping h^* is holomorphic and surjective.

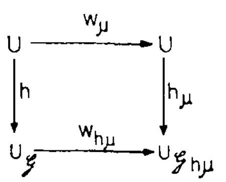

Fig. 1

The mapping h of (4) is a one-to-one mapping of the set of Beltrami coefficients for Γ that satisfy the Teichmüller condition onto the set of Beltrami coefficients for $\mathscr{G}$ that satisfy the Teichmüller condition. (These coefficients arise from integrable quadratic differentials for G that are allowed to have poles at the points $z_1, \cdots, z_n$.) Thus by the Teichmüller existence and uniqueness theorem (see, for example, Ahlfors [1] or Bers [5]), part (i) of Theorem 4.1 is established. (Another proof will follow from the method used for the general case outlined below.) With little effort we have just reproven the Bers-Greenberg isomorphism theorem [10] for finitely generated groups (this is also established by Marden [20]).

To continue with the general case, let

$$\operatorname{Cov} h = \{\mu \in M(\Gamma);\ h\mu \sim 0(\mathscr{G})\}.$$

It is easy to check that for each $\mu \in \operatorname{Cov} h$, there exists a $\beta \in \operatorname{Aut} U$ such that $\beta \circ w_\mu$ conjugates Γ into itself and thus determines an element $(\beta \circ w_\mu)^*$ of $\operatorname{Mod}\Gamma$ which is independent of the choice of β. The subgroup of all such elements of $\operatorname{Mod}\Gamma$ will be denoted by $\operatorname{Cov} h^*$. The group $\operatorname{Cov} h^*$ acts freely on $T(\Gamma)$; it is the covering group of h^*. Thus

$$T(\Gamma)/\mathrm{Cov}\, h^* \cong T(\mathscr{G}),$$

and hence Cov h^* is trivial if and only if $T(\mathscr{G})$ is simply connected.

To finish the proof of Theorem 4.1, we introduce the fiber spaces

$$\begin{aligned} F^n(G) &= \{(\tau,\zeta) \in T(G) \times \mathbb{C}^n;\ \tau = \Phi_G(\mu),\ \mu \in M(G), \\ \zeta &= (\zeta_1, \cdots, \zeta_n) \in w^\mu(U)^n\}, \end{aligned}$$

and

$$\begin{aligned} F_0^n(G) = \{(\tau,\zeta) \in F^n(G);\ \zeta = (\zeta_1, \cdots, \zeta_n) \in w^\mu(U)^n, \text{ where} \\ \zeta_j \text{ is not an elliptic fixed point of } G^\mu, \text{ for } j = 1, \cdots, n, \text{ and} \\ \zeta_i \neq g(\zeta_j) \text{ for all } g \in G^\mu \text{ whenever } i \neq j,\ i,j = 1, \cdots, n\}. \end{aligned}$$

REMARK. The fiber spaces $F^n(G)$ and $F_0^n(G)$ depend only on the signature of G. Hence $F(g; \nu_1, \cdots, \nu_m)$ is defined for signature $(g; \nu_1, \cdots, \nu_m)$ with $\nu_j > 1$ all j.

LEMMA 4.2. *For any* $\mathscr{G} = \{G; z_1, \cdots, z_n\}$, *we have*

$$T(\mathscr{G}) \cong F_0^n(G).$$

Since $F^1(G) = F_0^1(G)$ is simply connected for G without torsion, part (ii) has been established. Furthermore, the group Cov h^* is isomorphic to the fundamental group of $F_0^n(G)$ which is topologically a product

$$T(G) \times (U_G^n)_0,$$

where

$$\begin{aligned} (U_G^n)_0 = \{\zeta = (\zeta_1, \cdots, \zeta_n) \in U_G^n; \\ \zeta_i \neq g(\zeta_j) \text{ for all } g \in G,\ i \neq j,\ i,j = 1, \cdots, n\}. \end{aligned}$$

The two special cases discussed above give the isomorphism theorems of Bers-Greenberg [10] and Bers [9].

COROLLARY 4.3. *If* G_1 *and* G_2 *are Fuchsian groups of the first kind which are finitely generated, then* $T(G_1) \cong T(G_2)$ *whenever* $U_{G_1}/G_1 \cong U_{G_2}/G_2$. *(The Teichmüller spaces of two groups of type* $(g, 0, m)$ *are isomorphic.)*

COROLLARY 4.4. *If* G *and* G_1 *are Fuchsian groups of the first kind which are finitely generated without torsion such that* U/G *is conformally equivalent to* U/G_1 *punctured at one point, then*

$$T(G) \cong F(G_1) = F^1(G_1).$$

Restated,

$$T(g;\underbrace{\infty,\cdots,\infty}_{(m+1)\text{-times}}) \cong F(g;\underbrace{\infty,\cdots,\infty}_{m\text{-times}}).$$

COROLLARY 4.5. *If $\mathcal{G}$ is of type $(g,n,m-n)$, then $T(\mathcal{G})$ is a complex analytic manifold of dimension $3g-3+m$; in fact it is a bounded submanifold of* C^{3g-3+m}.

To describe $T(\mathcal{G},K)$, we must study the modular group, $\operatorname{Mod}\mathcal{G}$.

Every quasiconformal automorphism W that conjugates $\mathcal{G}$ into itself, fixes $U_{\mathcal{G}}$, and hence can be lifted to an automorphism w of U that conjugates Γ into itself. The relation between W and w is given by

$$h \circ w = W \circ h.$$

This element w induces a point of $\operatorname{Mod}\Gamma$. Let us denote by $\operatorname{Mod}(\Gamma,\mathcal{G})$ the subgroup of $\operatorname{Mod}\Gamma$ so produced. We then have the exact sequence

$$\begin{array}{c}\{1\}\to \operatorname{Cov} h^* \to \operatorname{Mod}(\Gamma,\mathcal{G}) \to \operatorname{Mod}\mathcal{G} \to \{1\},\\ \cap \\ \operatorname{Mod}\Gamma\end{array}$$

and complex analytic isomorphisms

$$T(\Gamma)/\operatorname{Cov} h^* \cong T(\mathcal{G}),$$

$$T(\Gamma)/\operatorname{Mod}(\Gamma,\mathcal{G}) \cong T(\mathcal{G})/\operatorname{Mod}\mathcal{G}.$$

Furthermore, $\operatorname{Mod}(\Gamma,\mathcal{G})$ is of finite index in $\operatorname{Mod}\Gamma$. To compute the index of $\operatorname{Mod}(\Gamma,\mathcal{G})$ in $\operatorname{Mod}\Gamma$ we consider only the *non-exceptional* cases (for the sake of brevity); that is, those cases where the signature of neither Γ nor $\mathcal{G}$ appear in the table in the first remark of Section 5. Let m be the number of punctures in $U_{\mathcal{G}}/G$. Call two punctures *equivalent* if the stability subgroups of their preimages in U are isomorphic. Let P_m be the permutation group on m letters, and $P_{0,m}$ the subgroup of P_m that preserves each equivalence class.

PROPOSITION 4.6. *In the non-exceptional cases there exists a surjective group homomorphism*

$$\theta\colon \operatorname{Mod}\Gamma \to P_m,$$

such that

$$\theta^{-1}(P_{0,m}) = \operatorname{Mod}(\Gamma,\mathcal{G}).$$

Let $\mathcal{G}$ have signature $(g;\nu_1,\cdots,\nu_m)$, and define for each l $(1 \leqq l \leqq \infty)$,

$$\alpha_l = \operatorname{card}\{j;\, \nu_j = l\}.$$

COROLLARY 4.7. *The index of* $\mathrm{Mod}(\Gamma, \mathscr{G})$ *in* $\mathrm{Mod}\,\Gamma$ *is*

$$\left(\sum_{l=1}^{\infty} \alpha_l\right)! \Big/ \prod_{l=1}^{\infty} \alpha_l!.$$

To study the group $\mathrm{Mod}\,\mathscr{G}$, let $G(n)$ denote the twisted product of P_n and G^n, where we define

$$(\sigma; g_1, \cdots, g_n) \cdot (\tau; \gamma_1, \cdots, \gamma_n) = (\sigma\tau; g_1 \circ \gamma_{\sigma^{-1}(1)}, \cdots, g_n \circ \gamma_{\sigma^{-1}(n)}),$$

$$\text{for } \sigma, \tau \in P_n,\ g_j, \gamma_i \in G,\ i, j = 1, \cdots, n.$$

The group $G(n)$ acts as a group of holomorphic automorphisms on U^n (also $(U_G^n)_0$) by

$$(\sigma; g_1, \cdots, g_n)(\zeta_1, \cdots, \zeta_n) = (g_1(\zeta_{\sigma^{-1}(1)}), \cdots, g_n(\zeta_{\sigma^{-1}(n)})).$$

We let $N_0(G)$ denote those elements β in $N(G)$, the normalizer of G in $\mathrm{Aut}\ U$, such that $\beta^* = 1$ as an element of $\mathrm{Mod}\,G$. The group G is then a normal subgroup of $N_0(G)$. We consider G as a subgroup of $G(n)$ via the inclusion

$$G \ni g \mapsto (1; g, \cdots, g) \in G(n).$$

We let $N_0(G)$ act on U^n (also on $(U_G^n)_0$) via

$$g(\zeta_1, \cdots, \zeta_n) = (g\zeta_1, \cdots, g\zeta_n), \text{ for } g \in N_0(G),$$

and observe that this action is compatible with the inclusion of G into $G(n)$. Let $\tilde{G}(n)$ be the smallest subgroup of the group of complex analytic automorphisms of U^n generated by $N_0(G)$ and $G(n)$. We observe that every element of $\tilde{G}(n)$ can be written (not uniquely) as

$$h\Sigma, \text{ with } h \in N_0(G),\ \Sigma \in G(n).$$

If $\Sigma = (\sigma; g_1, \cdots, g_n)$, then

$$h\Sigma = (\sigma; h \circ g_1, \cdots, h \circ g_n),$$

and

$$\Sigma h = (\sigma; h \circ g_1^h, \cdots, h \circ g_n^h),$$

where

$$g^h = h^{-1} \circ g \circ h, \text{ for } h \in N_0(G),\ g \in G.$$

Note also that we have the exact sequence of groups and group homomorphisms

$$\{1\} \to G(n) \to \tilde{G}(n) \to N_0(G)/G \to \{1\}.$$

THEOREM 4.8. *There is an exact sequence of groups and group homomorphisms of the form*

$$\{1\} \to \tilde{G}(n) \xrightarrow{\textcircled{H}_1} \operatorname{Mod}\mathcal{G} \xrightarrow{\textcircled{H}_2} \operatorname{Mod} G \to \{1\}.$$

Again, we only outline the construction of the various homomorphisms. For each

$$\Sigma = (\sigma; g_1, \cdots, g_n) \in G(n),$$

select $w = w_{\mu(\Sigma)}$ such that

$$\mu(\Sigma) \sim 0(G),$$

and

$$w(z_j) = g_{\sigma(j)}^{-1}(z_{\sigma(j)}), \text{ for } j = 1, \cdots, n.$$

Let $\textcircled{H}_1\Sigma = w^*$ (as an element of Mod $\mathcal{G}$).

Next if $g \in N_0(G)$, choose $\mu = \mu(g) \in M(G)$ such that $\mu(g) \sim 0(G)$ and $w = g \circ \omega_\mu$ satisfies

$$w(z_j) = z_j, \; j = 1, \cdots, n.$$

Set $\textcircled{H}_1 g = w^*$ as element of Mod $\mathcal{G}$. One must verify that $\textcircled{H}_1$ is well defined on $\tilde{G}(n)$.

To define $\textcircled{H}_2$ let w be a quasiconformal automorphism of U that conjugates $\mathcal{G}$ into itself. Since w conjugates G into itself, it induces an element of Mod G as well as of Mod $\mathcal{G}$. The surjectivity of $\textcircled{H}_2$ is obtained by observing that an arbitrary element of Mod G is induced by an automorphism that conjugates G into itself and fixes $z_1, \cdots, z_n$.

We next describe the action of Mod $\mathcal{G}$ on $F_0^n(G)$ via the isomorphism of Lemma 4.2. It is quite clear that in analogy to the previous definition, we have for each $\mu \in M(G)$ a group

$$\tilde{G}^\mu(n)$$

that acts holomorphically on

$$U^n_{G^\mu} = w^\mu(U^n_G).$$

Furthermore, both the group and the space on which it acts depend only on $\Phi_G(\mu) \in T(G)$. There is a natural isomorphism

$$\tilde{G}(n) \ni \Sigma \mapsto \Sigma^\mu \in \tilde{G}^\mu(n)$$

under which $G(n)$ is taken onto $G^\mu(n)$. This group isomorphism is compatible with the isomorphism of complex manifolds of Lemma 4.2, in the sense that for $\Sigma \in G(n)$, its action as an element of Mod $\mathcal{G}$ on $F^n(G)$ is given by

$$([w_\mu]_G, \zeta) \overset{\Sigma}{\mapsto} ([w_\mu]_G, \Sigma^\mu \zeta);$$

here $\mu \in M(G)$, $\zeta = (\zeta_1, \cdots, \zeta_n) \in w^\mu(U)^n$. Note also that $\Sigma(F_0^n(G)) = F_0^n(G)$.

Next we take an element $g \in N_0(G)$ and consider its action on $F^n(G)$. It is given by

$$(5) \qquad ([w_\mu]_G, \zeta) \overset{g}{\mapsto} ([w_\mu]_G, g^\mu \zeta) \text{ with } g^\mu = w^\mu \circ g \circ (w^\mu)^{-1}.$$

Again, $g(F_0^n(G)) = F_0^n(G)$.

Finally we examine the action on $F_0^n(G)$ of the subgroup of Mod $\mathcal{G}$ induced by quasiconformal automorphisms of U that are compatible with G, and fix $z = (z_1, \cdots, z_n)$. For such a w,

$$(6) \qquad ([w_\mu]_G, \zeta) \overset{w^*}{\mapsto} ([\alpha[\mu] \circ w_\mu \circ w^{-1}]_G, h_{*\mu} \circ \alpha[\mu] \circ h_\mu^{-1}(\zeta)),$$

where $\alpha[\mu] \in \operatorname{Aut} U$ is chosen so that $\alpha[\mu] \circ w_\mu \circ w^{-1}$ is normalized and hence $w_{*\mu}$ for some $*\mu \in M(G)$, and h_ν for $\nu \in M(G)$, is the unique conformal map so that

$$w^\nu = h_\nu \circ w_\nu.$$

Again, w^* acts on a $F_0^n(G)$ as well as $F^n(G)$.

Note that the group G^n is a subgroup of $G(n)$ and that it acts freely on $F_0^n(G)$. As a consequence of this observation we have Corollary 4.9.

COROLLARY 4.9. *Let $\mathcal{G}$ be an n-pointed Fuchsian group, $\mathcal{G} = \{G; z_1, \cdots, z_n\}$, and K a subgroup of G; then*

$$T(\mathcal{G}, K) \cong F_0^n(G)/K^n.$$

In particular, if $\mathcal{G}$ is of type $(g, n, m-n)$, then $T(\mathcal{G}, K)$ is a $3g-3+m$ dimensional complex manifold.

COROLLARY 4.10. *There is an exact sequence of groups and group homomorphisms of the form*

$$\{1\} \to G^n \to \operatorname{Mod} \mathcal{G} \to \operatorname{Mod}(\mathcal{G}, G) \to \{1\}.$$

A similar description may be obtained of the group $\operatorname{Mod}(\mathcal{G}, K)$ for arbitrary normal subgroups K of G. This group is a factor group of a subgroup of Mod $\mathcal{G}$.

Consider the special case where $\mathcal{G} = \{G; z\}$ and G is fixed point free. Then we have the following complex analitic projections

$$T(\mathscr{G}) \xrightarrow{\pi_2} T(\mathscr{G}, G) \xrightarrow{\pi_1} T(G).$$

Note that $T(\mathscr{G}, G)$ is the universal curve over the Teichmüller space $T(G)$, ($\pi_1^{-1}(\tau)$ is the Riemann surface represented by τ or $\pi_1^{-1}(\tau) = \tau$), and $T(\mathscr{G})$ is the universal covering space of the universal curve.

REMARK. Assume that $[w_\mu]_G$ is a fixed point of w^* (as an element of Mod G) in (6). Thus

$$[w_\mu]_G = [\alpha[\mu] \circ w_\mu \circ w^{-1}]_G,$$

or

$$h_{*\mu} = h_\mu.$$

In this case $\alpha[\mu] \in N(G_\mu)$ and hence $\alpha = h_\mu \circ \alpha[\mu] \circ h_\mu{}^{-1} \in N(G^\mu)$.

Conversely, for each $\alpha \in N(G^\mu)$, there exists a $w^* \in \operatorname{Mod} G$ such that $[w_\mu]_G$ is a fixed point of $\operatorname{Mod} G$; the action of w^* as an element of $\operatorname{Mod} \mathscr{G}$ on the fiber over $[w_\mu]_G$ is precisely (assuming $w(z) = z$)

$$w^*([w_\mu]_G, \zeta) = ([w_\mu]_G, \alpha\zeta) \text{ for } \zeta \in w^\mu(U^n).$$

5. Riemann spaces

Let, as before,

$$\mathscr{G} = \{G; z_1, \cdots, z_n\}$$

be an n-pointed Fuchsian group. We define the *Riemann space* of $\mathscr{G}$ by

$$R(\mathscr{G}) = T(\mathscr{G})/\operatorname{Mod} \mathscr{G}.$$

Since $\operatorname{Mod} \mathscr{G}$ acts discontinuously on $T(\mathscr{G})$, it follows from a theorem of Cartan [13] that $R(\mathscr{G})$ is a normal complex space. $R(\mathscr{G})$ represents the set of conjugacy classes of n-pointed Fuchsian groups. It depends only on the signature of $\mathscr{G}$. Hence we may define

$$R(g; \nu_1, \cdots, \nu_m)$$

as $R(\mathscr{G})$ for some group $\mathscr{G}$ of signature $(g; \nu_1, \cdots, \nu_m)$.

Let Γ be the Fuchsian model of $(G, U_{\mathscr{G}})$. Consider the complex analytic varieties and mappings shown in Fig. 2, where $\tilde{G}(n) \otimes \operatorname{Mod} G$ denotes the exact sequence decomposition of $\operatorname{Mod} \mathscr{G}$ obtained by Theorem 4.8.

We describe each map in Fig. 2.

(i) The map π_1 is the finite-sheeted surjective covering induced by the inclusion $\operatorname{Mod}(\Gamma, \mathscr{G}) \subset \operatorname{Mod} \Gamma$. This map is injective if and only if $U_{\mathscr{G}}/G$ has one equivalence class of punctures. Thus π_1 is the identity map only if $\mathscr{G}$ has signature $(g; \nu, \cdots, \nu)$.

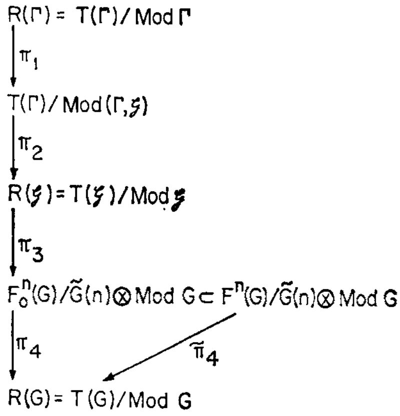

Fig. 2

(ii) The map π_2 is the canonical isomorphism induced by Theorem 4.1.

(iii) The map π_3 is the canonical isomorphism induced by the isomorphisms of Lemma 4.2 and Theorem 4.8.

(iv) The map $\tilde{\pi}_4$ is induced by the natural projection onto the first coordinate

$$F^n(G)/\tilde{G}(n) \xrightarrow{\rho} T(G).$$

The fiber over a point $\tau = \Phi_G(\mu) \in T(G)$ (with $\mu \in M(G)$) is the complex analytic space $w^\mu(U^n)/\tilde{G}^\mu(n)$. This is nothing more than the n-dimensional symmetric product of $w^\mu(U)/G^\mu$ factored by the group $N_0(G^\mu)/G^\mu$. The action of $N_0(G)^\mu/G^\mu$ on the symmetric product is derived from (5).

Remark. The group $N_0(G)/G$ is trivial except where G has one of the following signatures (see, for example, Singerman [23]):

$$
\begin{array}{ll}
(2;\, -) & \\
(1;\, \nu, \nu) \quad , & \nu \geqq 2, \\
(1;\, \nu) \quad , & \nu \geqq 2, \\
(0;\, \nu, \nu, \mu, \mu)\, , & 2 \leqq \nu \leqq \mu \leqq \infty, \\
(0;\, \nu, \nu, \mu) \quad , & 2 \leqq \nu \leqq \mu \leqq \infty, \\
(0;\, \nu, \mu, \mu) \quad , & 2 \leqq \nu < \mu \leqq \infty.
\end{array}
$$

In all cases the group is finite with an index less than or equal to 6.

We continue with the description of the mapping $\tilde{\pi}_4$. We describe the fiber over a point $\tau \in R(G)$. If τ is the equivalence class of $[w_\mu]_G \in T(G)$ (with $\mu \in M(G)$), then the fiber over τ is the n-fold symmetric product of $w^\mu(U)/G^\mu$ factored by $N(G^\mu)/G^\mu$. (If G has no torsion this is the full analytic automorphism group of $w^\mu(U)/G^\mu$.)

The map π_4 is, of course, the restriction of $\tilde{\pi}_4$ to $F_0^n(G)/\tilde{G}(n) \otimes \mathrm{Mod}\, G$. The fiber over τ, in this case, is the n-dimensional symmetric product of $w^\mu(U)/G^n$ with the thick diagonal deleted, factored by $N(G^\mu)/G^\mu$.

The above fibers can also be interpreted as positive divisors of degree n on various Riemann surfaces.

In the special case where $n = 1$, we obtain a result of Bers [9].

THEOREM 5.1. *Let $(g; v_1, \cdots, v_m)$ be a signature of a Fuchsian group. There exists a holomorphic surjection*

$$\pi: R(g; 1, v_1, \cdots, v_m) \to R(g; v_1, \cdots, v_m),$$

such that for $\tau \in R(g, v_1, \cdots, v_m)$, $\pi^{-1}(\tau)$ is the reduced surface corresponding to τ. Furthermore, there exists a finitely sheeted holomorphic surjection

$$\rho: R(g; \underbrace{\infty, \cdots, \infty}_{(m+1)\text{-times}}) \to R(g; 1, v_1, \cdots, v_m).$$

In particular, for signature $(g; \underbrace{\infty, \cdots, \infty}_{m\text{-times}})$, ρ is $m + 1$ sheeted.

If G is any Fuchsian group then, by the reduced surface corresponding to G, we mean $U_G/N(G)$. Note that the conformal type of the *reduced surface* depends only on the conjugacy class of G in Aut U.

REMARK. For groups G of signature $(g; -)$with $g \geqq 2$, an equivalent theorem was announced by Teichmüller [26] and first proven by Baily [4].

Another interesting special case of the above construction is contained in Theorem 5.2.

THEOREM 5.2. *For every $g \geqq 2$, there exists a holomorphic surjection*

$$\pi: R(g; \underbrace{\infty, \cdots, \infty}_{m\text{-times}}) \to R(g; \ -)$$

such that, for rigid $\tau \in R(g\ ;\ -\ )$, $\pi^{-1}(\tau)$ is the m-fold symmetric product of τ with the thick diagonal deleted.

It is easy to describe geometrically the mapping π of Theorem 5.2. A point

$x \in R(g; \underbrace{\infty, \cdots, \infty}_{m\text{-times}}\}$ represents generically a compact surface of genus g with m punctures. The mapping π forgets the punctures and thus $\pi(x) \in R(g; -)$ represents a compact surface of genus g.

6. Homotopy and isotopy classes of self-mappings of surfaces

Let S be a surface of finite type; that is, S is topologically equivalent to a compact Riemann surface of genus $g \geqq 0$ with $(m - n) \geqq 0$ punctures. We assume that

$$2g + (m - n) \geqq 3.$$

(This excludes certain cases that can be handled by different and simpler methods.) Fix $n \geqq 0$ distinct points on S: $x_1, \cdots, x_n$.

DEFINITION 6.1. Let $\mathscr{H}(g, m - n, n)$ denote the group of all orientation-preserving homeomorphisms of S onto itself which are homotopic to the identity by a homotopy keeping $x_1, \cdots, x_n$ fixed, modulo those homeomorphisms that are homotopic to the identity on $S' = S - \{x_1, \cdots, x_n\}$.

Applying our previous work, let us obtain a description of $\mathscr{H}(g, m - n, n)$. It suffices (by a theorem of Bers [6]) to consider only quasiconformal automorphisms w of S.

Let $\mathscr{G} = \{G; z_1, \cdots, z_n\}$ be an n-pointed Fuchsian group. Let h and Γ have the same meaning as in Section 4. Consider the group

$$\mathscr{H}(\mathscr{G}) = \{\mu \in M(G); \mu \sim 0(\mathscr{G})\} / \{\mu \in M(G); h^{-1}\mu \sim 0(\Gamma)\}.$$

PROPOSITION 6.2. *If G has signature $(g; \underbrace{\infty, \cdots, \infty}_{(m-n)\text{-times}})$, and $\mathscr{G} = \{G; z_1, \cdots, z_n\}$ is an n-pointed Fuchsian group, then $\mathscr{H}(\mathscr{G}) \cong \mathscr{H}(g, m - n, n)$.*

THEOREM 6.3. *We have for an arbitrary n-pointed Fuchsian group,*

$$\mathscr{H}(\mathscr{G}) \cong \operatorname{Cov} h^* \cong \pi_1(T(\mathscr{G})) \cong \pi_1(U_G^{n\#}),$$

where, as usual, $\pi_1(-)$ is the fundamental group of $(-)$, and

$$U_G^{n\#} = \{z = (z_1, \cdots, z_n) \in U_G^n;\ z_i \neq g(z_j) \text{ all } g \in G \text{ for } i \neq j\}.$$

COROLLARY 6.4. (i) (Epstein [18]). *The group $(g, m - 1, 1)$ is trivial.*

(ii) *The group $(g, m - 2, 2)$ is a free group on countably many generators.*

Part (i) of Corollary 6.4 has also been obtained by Bers [9]. Part (ii) is obtained by considering the locally trivial fibration which is a projection on first $(n-1)$ coordinates, as shown in Fig. 3, where the fiber U' is U punctured at countably many points. This fibration induces a long exact sequence of fundamental groups that contains the portion

$$\to \pi_i(U') \to \pi_i(U_G^{n\#}) \to \pi_i(U_G^{(n-1)\#}) \to \pi_{i-1}(U') \to \cdots.$$

Since U' and $U_G^{n\#}$ have contractible universal covering spaces,

$$\pi_i(U') = \{0\} = \pi_i(U_G^{n\#}),\ i \geqq 2,\ n \geqq 1.$$

Thus the previous exact sequence for $i = 1$ reduces to

$$\{0\} \to \pi_1(U') \to \pi_1(U_G^{n\#}) \to \pi_1(U_G^{(n-1)\#}) \to \{0\}.$$

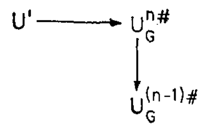

Fig. 3

Our result can be strengthened slightly. Let

$$M_0(\mathscr{G}) = \{\mu \in M(G);\ \mu \sim 0(\mathscr{G})\}.$$

We again have a locally trivial fibration:

$$\begin{array}{ccc} M_0(\mathscr{G}) & \to & M(G) \\ & & \downarrow \\ & & T(\mathscr{G}). \end{array}$$

Using exact sequences as above, we obtain Theorem 6.5.

THEOREM 6.5. *We have*

$$\pi_i(M_0(\mathscr{G})) = \{0\},\ i \geqq 1,$$
$$\pi_0(M_0(\mathscr{G})) \cong \pi_1(T(\mathscr{G})).$$

COROLLARY 6.6. *We have*

$$\mathscr{H}(g, m-n, n) \cong \frac{\text{self-mappings of } S \text{ homotopic to the identity modulo } (x_1, \cdots, x_n)}{\text{self-mappings of } S \text{ isotopic to the identity modulo } \{x_1, \cdots, x_n\}}.$$

COROLLARY 6.7. *The group $M_0(\mathscr{G})$ is contractible if and only if $n = 0$ or G is torsion free and $n = 1$.*

The above theorem and its corollaries are similar to results of Earle and Eells [15].

7. Problems

The theory outlined in the course of this paper presents more interesting problems than those we have solved. We present a few of them in the concluding section.

PROBLEM 7.1. Classify up to biholomorphic equivalence the spaces

$$T(g; \nu_1, \cdots, \nu_m).$$

If $\nu_j \geqq 2$ for all j, then such a classification (as a consequence of Bers-Greenberg [10]) was obtained by Patterson [21].

PROBLEM 7.2. Classify up to biholomorphic equivalence the spaces

$$F^n(g; \nu_1, \cdots, \nu_m) \text{ and } F_0^n(g; \nu_1, \cdots, \nu_m).$$

Here, of course, $\nu_j \geqq 2$ for all j. A partial result for $n = 1$ is contained in Earle-Kra [16].

PROBLEM 7.3. What is the Kobayashi metric on $F^n(g; \nu_1, \cdots, \nu_m)$?

For $T(g; \nu_1, \cdots, \nu_m)$ with $\nu_j \geqq 2$ for all j, Royden [22] showed that the Teichmüller and Kobayashi metrics agree. (See also Earle-Kra [16].)

PROBLEM 7.4. Describe the automorphism group of $F^n(g; \nu_1, \cdots, \nu_m)$.

A slightly more accessible problem might be to describe those automorphisms of $F^n(g; \nu_1, \cdots, \nu_m)$ that are invariant under the natural projection

$$\pi: F^n(g; \nu_1, \cdots, \nu_m) \to T(g; \nu_1, \cdots, \nu_m);$$

that is, those f for which the diagram shown in Fig. 4 is commutative.

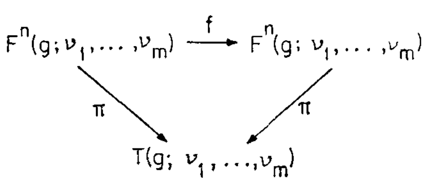

Fig. 4

The description of the automorphism group of $T(g; \nu_1, \cdots, \nu_m)$ with $\nu_j \geqq 2$, has been obtained by Royden [22]. (See also Earle-Kra [16].)

PROBLEM 7.5. Classify all complex spaces X lying above $R(g; \nu_1, \cdots, \nu_m)$ and below $T(g; \nu_1, \cdots, \nu_m)$; that is, all X for which the diagram shown in Fig. 5 is commutative. (The maps ρ_1, ρ_2, π are, of course, analytic.)

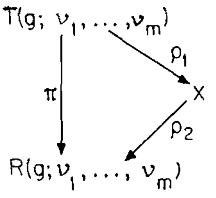

Fig. 5

PROBLEM 7.6. Determine all holomorphic cross sections

$$s: T(g; \nu_1, \cdots, \nu_m) \to F^n(g; \nu_1, \cdots, \nu_m)$$

of the holomorphic projection

$$\pi: F^n(g; \nu_1, \cdots, \nu_m) \to T(g; \nu_1, \cdots, \nu_m);$$

that is, all s as above with

$$\pi \circ s = \text{id}.$$

For $n = 1$, a partial solution has been obtained by Hubbard [19] and Earle-Kra [16].

ADDED IN PROOF

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